Multiple and least energy sign-changing solutions for Schr¨odinger-Poisson equations in R3 with restraint

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ABSTRAC

In this paper, we study the existence of multiple sign-changing solutions with a prescribed L^{p+1}-norm and the existence of least energy sign-changing restrained solutions for the following nonlinear Schrödinger-Poisson system:

$$\begin{cases} -\Delta u + u + \phi(x)u = \lambda |u|^{p-1} u, & \text{in } \mathbb{R}^3, \\ -\Delta\phi(x) = |u|^2, & on\ \mathbb{R}^3. \end{cases}$$

By choosing a proper functional restricted on some appropriate subset to using a method of invariant sets of descending flow, we prove that this system has infinitely many sign-changing solutions With the prescribed L^{p+1}-norm and has a least energy for such sign-changing restrained solution for $p \in (3,5)$. Few existence results of multiple sign-changing restrained solutions are available in the literature. Our work generalize some results in literature.

Indexing terms/Keywords

sign-changing solution, prescribed L^{p+1}-norm, multiplicity, local genus.

Academic Discipline And Sub-Disciplines

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SUBJECT CLASSIFICATION

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TYPE (METHOD/APPROACH)

Nonlinear analysis, critical points theory, variational method.

1. INTRODUCTION AND MAIN RESULTS

In this paper, we study the multiplicity of sign-changing solutions of the following nonlinear Schrödinger-Poisson system:

$$\begin{cases} -\Delta u + u + \phi(x)u = \lambda |u|^{p-1} u, & \text{in } \mathbb{R}^3, \\ -\Delta\phi(x) = |u|^2, & \text{on } \mathbb{R}^3. \end{cases} \tag{1.1}$$

where $p \in (3,5)$, $\lambda \in \mathbb{R}$ is a parameter. This system has been first introduced in [1] as a physical model describing a charged wave interacting with its own electrostatic field in quantum mechanic. The unknowns of the system are the field u associated to the particle and the electric potential ϕ. The presence of the nonlinear term simulates the interaction between many particles or external nonlinear perturbations. We refer the readers to [1] and the references therein for the physical aspects of problem (1.1). Similar equations have been very studied in literature, see [2-7,10-16].

The $\lambda \in \mathbb{R}$ in (1.1) is called a frequency. For fixed λ, system (1.1) has been extensively studied on the existence of positive solutions, ground states, radial and non-radial solutions and semiclassical states, see e.g. [6-17], etc. As shown by recent results the structure of the solution set of (1.1) depends strongly on the value of p of the power-type nonlinearity. In [6] and [8], a related Pohozeav equality is found, and then the authors proved that system (1.1) does not admit any nontrivial solution for $p \le 2$ or $p \ge 5$ if $\lambda = 1$. While as $p \in (2,5)$, the existence and multiplicity results have been obtained for $\lambda > 0$ by using variational techniques.

To continue the statement well, let us fix some notations. We will write $H^1 = H^1(\mathbb{R}^3)$, $D^1 = D^{1,2}(\mathbb{R}^3) = \{u \in L^6(\mathbb{R}^3) : \nabla u \in$

$L^2(\mathbb{R}^3)\}$ as the usual Sobolev spaces, and H_r^1, D_r^1 the corresponding subspaces of radial functions. Recall that the inclusion $H_r^1 \to L^q = L^q(\mathbb{R}^3)$ is compact for $2<q<6$ (see [18]). In the present paper, we will take $H = H_r^1$ as the work space. Sometimes we will simply write $\int f$ to mean the Lebesgue integral of $f(x)$ in $\mathbb{R}^3$. We make use of the following notations.

$$|u|_p = (\int_{\mathbb{R}^3} |u(x)|^p\, dx)^{\frac{1}{p}} \text{ for } p\in[2,+\infty) \text{ and } u\in L^p;$$
$$\|u\| = [\int_{\mathbb{R}^3} |\nabla u|^2\, dx + \int_{\mathbb{R}^3} |u|^2\, dx]^{\frac{1}{2}} \text{ for } u \in H^1 = H^1(\mathbb{R}^3).$$

c, d, c_j, d_j Denote positive constants which can change line to line.

We say that $(u_c, \lambda_c) \in H_r^1(\mathbb{R}^3)\times\mathbb{R}$ is a couple of solution to (1.1) if u_c is a solution to (1.1) with $\lambda = \lambda_c$. Motivated by the fact that physicists are often interested in restrained solutions or normalized solutions, that is, solutions with a prescribed L^{p+1}-norm, we consider for each $c>0$ the following problem:

($\mathrm{P_c}$): There exists a couple $(u_c, \lambda_c) \in H_r^1(\mathbb{R}^3)\times\mathbb{R}$ of solution to (1.1) such that $|u|_{p+1}^{p+1} = c$.

Recently, normalized or restrained solutions to elliptic equations attract much attention of researchers, see e.g. [19-31]. In [19], Liu and Wang considered the restrained problem to the following quasilinear Schrödinger equation:

$$-\Delta u + V(x)u - \frac{1}{2}u\Delta u^2 = \lambda |u|^{p-1} u \qquad \text{in } \mathbb{R}^N. \tag{1.2}$$

They proved the existence of a positive solution with the restraint $\int_{\mathbb{R}^N} |u|^{p+1}\, dx = 1$, and λ appears as an unknown Lagrangian multiplier, to Eq. (1.2). In [20], Xiong and Liu proved the existence of a sign-changing solution with the restraint $|u|_{p+1}^{p+1} = 1$ to (1.2). In [21], Benci and Cerami considered the following semi-linear Schrödinger equation:

$$-\Delta u - \lambda u = g(u), \qquad \lambda\in\mathbb{R}, \quad x\in\mathbb{R}^N. \tag{1.3}$$

With $g(u) = |u|^{p-1}u$, they proved the existence of multiple positive solutions with the restraint $|u|_{p+1}^{p+1} = 1$ to (1.3).

In [23], by using a minimax procedure, Jeanjean proved that for each $c>0$, there is a couple $(u_c, \lambda_c) \in H^1(\mathbb{R}^N) \times\mathbb{R}^-$ of weak solution to (1.3) with $|u|_2^2 = c$. In [26], Bartsch and De Valeriola considered the semi-linear Schrödinger equation (1.3) and proved that there are infinitely many normalized solutions to Eq. (1.3). In [27], Bellazzini et al. considered (1.1) and proved that for $p\in(\frac{7}{3}, 5)$ there exists $c_0>0$ such that for any $c\in(0, c_0)$, equation (1.1) has a couple $(u_c, \lambda_c)\in H^1(\mathbb{R}^N)\times\mathbb{R}^-$ of weak solution with $|u|_2^2 = c$ by using a mountain pass argument on

$$S(c) = \{u\in H^1(\mathbb{R}^3) : u|_2^2 = c\}, \qquad c>0.$$

Luo in [30] proved that when $p\in(\frac{7}{3}, 5)$, there exists $c_0>0$ such that for any $c\in(0,c_0)$, equation (1.1) admits an unbounded sequence of couples of weak solutions $\{(\pm u_n, \lambda_n)\}\subset H_r^1(\mathbb{R}^N)\times\mathbb{R}^-$ with $|u_n|_2^2 = c$ for each $n\in\mathbb{N}^+$. Luo and Wang in [31] proved that there are infinitely many normalized high energy solutions to Kirchhoff-type equations restrained on $S(c) = \{u\in H^1(\mathbb{R}^3) : u|_2^2 = c\}, c>0.$

On the other hand, the problem of finding sign-changing solutions is a very classical problem. In general, this problem is much more difficult than finding a mere solution. There were several abstract theories or methods to study sign-changing solutions. In recent years, for fixed λ, Wang and Zhou [32] obtained a least energy sign-changing solution to (1.1) without any symmetry by seeking minimizer of the energy functional on the sign-changing Nehari manifold when $p\in(3,5)$, based on variational method and Brouwer degree theory. Liu et al [33] considered a more general nonlinear term f, they proved that problem (1.1) has infinitely many sign-changing solutions under some appropriate conditions on the nonlinearity, especially, the f is quasi-asymptotic p order, i.e., $\limsup_{|s|\to+\infty} \frac{|f(s)|}{|s|^p} < +\infty$ for some $p\in(2,5)$. Using concentration compactness principle and rotational transformation, d' Aveni [34] showed the existence of non-radially symmetric sign-changing solution of (1.1). Using a Nehari type manifold and gluing solution piece together, Kim and Seok [35] proved the existence of radially sign-changing solutions of (1.1) with prescribed numbers of nodal domains for $p \in (3,5)$. Ianni [36] obtained a similar result to [35] for $p \in [3,5)$ via a heat flow approach together with a limit procedure. Based on the Lyapunov-Schmidt reduction method, in another paper of Ianni and Vaira [37], the existence of non-radially symmetric sign-changing solutions for the semi-classical limit case of (1.1).

Motivated by the above works, a natural question is whether (1.1) has sign-changing solutions u_c for problem (P_c) and whether (1.1) has infinitely many sign-changing restrained solutions u_c for problem (P_c). To the authors' knowledge, there are very few results on the multiple of sign-changing restrained solutions for problem (1.1) in the literature. In the present paper, we focus on the study of multiple sign-changing restrained solutions for system (1.1). We will verify that system (1.1) has infinitely many sign-changing restrained solutions for $p \in (3,5)$. Our main result in this aspect is the following:

Theorem 1.1. Let $p \in (3,5)$. Then for any given $c > 0$, equation (1.1) has a sequence of couples of sign-changing restrained solutions $\{(u_k, \lambda_k)\} \subset H_r^1(\mathbb{R}^3) \times \mathbb{R}^+$ with $|u_k|_{p+1}^{p+1} = c$ for each $k \in \mathbb{N}^+$.

To prove the theorem we use the general ideas inspired by [38] adapting their arguments to our problem which contains also the coupling term. Where a suitable subset was given in which there exist two subsets separating the motivating functional, and on which an auxiliary operator $\mathbf{A}$ was constructed, so that we are able to apply suitable minimax arguments in the presence of invariant sets of a descending flow generated by the operator $\mathbf{A}$ to obtain the existence of multiple sign-changing solutions with restraint to system (1.1). We have used this method to obtain an analogous result to (1.1) for $p \in (3,5)$ and $\lambda = 1$. Some arguments in our proof are borrowed from [38]. Remark that the ideas in [38] can not be used directly, and here we will give some new techniques. The method seems to be quite new for the nonlinear Schrödinger-Poisson equations and presents several difficulties due to nonlocal term. The method is different from that used in [20, 23, 26, 27] and others.

Since (1.1) has infinitely many sign-changing restrained solutions, another natural question is whether (1.1) has a least energy sign-changing restrained solution, which has not been studied before. Here we can prove the following result.

Theorem 1.2. Suppose that the conditions in Theorem 1.1 hold. Then system (1.1) has a least energy sign-changing solution (u_c, λ_c) with restraint $|u_c|_{p+1}^{p+1} = c$, that is, it has the least energy among all sign-changing radially solutions with restraint $|u_c|_{p+1}^{p+1} = c$.

The paper is organized as follows. In Section 2, we present some preliminary results. We prove Theorem 1.1 in section 3 and Theorem 1.2 in section 4, respectively.

2. PRELIMINARIES

In this section, we give some preliminary results. An important fact involving system (1.1) is that this class of system can be transformed into a Schrödinger equation with a nonlocal term (see, for instance, [8, 10]), which allows to apply variational approaches. For any given $u \in H^1$, the Lax-Milgram Theorem implies that there exists a unique $\Phi[u] = \phi_u \in D^1$ such that $-\Delta\phi = |u|^2$ and

$$\phi_u(x) = \int_{\mathbb{R}^3} \frac{u^2(y)}{4\pi|x-y|} dy.$$

We now summarize some properties of the map Φ, which will be useful later. See, for instance, [5] and [8] for a proof.

Lemma 2.1.

(1) The map $\Phi : u \in H^1 \to \phi_u \in D^1$ is of class C^1.

(2) $\Phi[u] = \phi_u \geq 0$.

(3) $\Phi[tu] = t^2 \Phi[u]$ for every $u \in H^1$ and $t \in \mathbb{R}$.

(4) There exists $c^* > 0$ independent of u such that

$$\int_{\mathbb{R}^3} \phi_u u^2 \leq c^* \|u\|^4 .$$

(5) If u is a radial function, then so is ϕ_u.

(6) If $u_n \to u$ weakly in H_r^1 then $\Phi[u_n] \to \Phi[u]$ in D_r^1, and $\int_{R^3} \Phi[u_n]u_n^2 \to \int_{R^3} \Phi[u]u^2$ strongly.

From above properties, substituting $\phi = \phi_u$ into system (1.1), we can rewrite system (1.1) as the single equation

$$-\Delta u + u + \phi_u u = \lambda |u|^{p-1} u,$$

and the energy functional $I_\lambda : H_r^1 \to \mathbb{R}$:

$$I_\lambda(u) = \int (\frac{1}{2}(|\nabla u(x)|^2 + |u(x)|^2) + \frac{1}{4}\phi_u(x)u^2(x) - \frac{1}{p+1}\lambda |u(x)|^{p+1})dx$$

is well defined for any $\lambda > 0$. Furthermore, it is known that I_λ is a C^1 functional with derivative given by

$$I'_\lambda(u)[v] = \int \nabla u \nabla v + uv + \phi_u uv - \lambda |u|^{p-1} uv.$$

Throughout this paper, we take the following functional

$$I(u) = \int (\frac{1}{2}(|\nabla u(x)|^2 + |u(x)|^2) + \frac{1}{4}\phi_u(x)u^2(x) - \frac{1}{p+1}|u(x)|^{p+1})dx \qquad (2.1)$$

as our motivating functional. However the functional is unbounded from above and from below on H_r^1. The idea is to restrict the functional to a suitable subset on which this unboundedness is removed, and in which we can select two subsets separating the motivating functional.

Define

$$\begin{aligned} M^* &= \{u \in H_r^1 : \frac{1}{2}c < |u|_{p+1}^{p+1} < 2c\}; \\ M &= \{u \in H_r^1 : |u|_{p+1}^{p+1} = c\}. \end{aligned}$$

Evidently M^* is open subset of H and M is closed. Define

$$N_b^* = \{u \in M^* : \|u\|^2 < b\}, \quad N_b = N_b^* \cap M.$$

We will see that, to obtain solutions of (1.1) solving problem (P_c), we turn to study the functional I restricted to N_b^*, which is a problem with another extra constraint. We obtain directly the couple on (u_c, λ_c) with restraint $|u_c|_{p+1}^{p+1} = c$ solving Eq. (1.1) without utilizing critical points of the functional I_λ. Recalling the Sobolev inequality

$$\|u\|^2 \geq S|u|_{p+1}^2, \qquad \forall u \in H_r^1,$$

where S is a positive constant.

Fix any $k \in \mathbb{N}$. Let W_{k+1} be a $k+1$ dimensional subspace of H. Then we can find some $b_k > 0$ such that

$$\|u\|^2 \leq b_k, \quad \forall u \in W_{k+1} \text{ satisfying } |u|_{p+1}^{p+1} < 2c. \qquad (2.2)$$

Fix a $b > 0$ such that

$$b > 2(\frac{1}{2}b_k + \frac{1}{4}c^* b_k^2 + \frac{c}{p+1} + 1) > b_k. \qquad (2.3)$$

From now on, we let $\alpha = \int (|\nabla u|^2 + |u|^2) = \|u\|^2$, $\beta = \int \phi_u(x)u^2$, $\gamma = \int |u(x)|^{p+1}$ as fixed notations for convenience. Let $B_k = \{u \in W_{k+1} : |u|_{p+1}^{p+1} = c\}$, and for any $u \in B_k$ we have that

$$I(u) = \frac{1}{2}\alpha + \frac{1}{4}\beta - \frac{1}{p+1}\gamma \leq \frac{1}{2}\alpha + \frac{1}{4}\beta.$$

Since $B_k \subset N_{b_k}$, we have that

$$\alpha = \| u \|^2 \le b_k, \quad \beta \le c * b_k^2.$$

Then we obtain that

$$I(u) \le \frac{1}{2} b_k + \frac{1}{4} c * b_k^2.$$

Let $d_k = \frac{1}{2} b_k + \frac{1}{4} c * b_k^2 + 1,$ therefore, we have

$$\sup_{u \in B_k} I(u) < d_k. \tag{2.4}$$

Then for $u \in N_b$, we have that

$$I(u) = \frac{1}{2}\alpha + \frac{1}{4}\beta - \frac{1}{p+1}\gamma \ge \frac{1}{2}\alpha - \frac{c}{p+1}.$$

And we have that

$$\inf_{u \in \partial N_b} I(u) \ge d_k. \tag{2.5}$$

Hence we achieve the following important lemma.

Lemma 2.2. There exists $d_k > 0$ such that

$$\inf_{u \in \partial N_b} I(u) \ge d_k > \sup_{u \in B_k} I(u). \tag{2.6}$$

Now we introduce an auxiliary operator $\mathbf{A}$, which will be used to construct the descending flow for the functional I. Clearly, for any $u \in N_b^*$, the operator $-\Delta + 1 + \phi_u$ is positive definite in H_r^1. For any $u \in N_b^*$, let $\tilde{w} \in H_r^1$ be the unique solution to the following linear equation

$$-\Delta \tilde{w} + \tilde{w} + \phi_u \tilde{w} = |u|^{p-1} u, \qquad \tilde{w} \in H_r^1. \tag{2.7}$$

Since $|u|_{p+1}^{p+1} > \frac{1}{2} c > 0,$, so $\tilde{w} \ne 0$ and

$$\int |u|^{p-1} u \tilde{w} = \| \tilde{w} \|^2 + \int \phi_u |\tilde{w}|^2 \ge \| \tilde{w} \|^2 > 0.$$

Let

$$w = \sigma \tilde{w}, \qquad \text{where} \quad \sigma = \frac{c}{\int |u|^{p-1} u \tilde{w}} > 0.$$

Then w is the unique solution of the following problem

$$\begin{cases} -\Delta w + w + \phi_u w = \sigma |u|^{p-1} u, \\ \int |u|^{p-1} u w = c, \qquad w \in H_r^1. \end{cases} \tag{2.8}$$

Then, the operator $\mathbf{A}$ is defined as follows: for any $u \in N_b^*, \mathbf{A}(u) = w \in H_r^1.$ Clearly, $\mathbf{A}$ is odd. Furthermore, we have

Lemma 2.3. The operator $\mathbf{A}$ is of class $\boldsymbol{C}^1$ from N_b^* to $H_r^1,$ that is, $\mathbf{A} \in \boldsymbol{C}^1 (N_b^*, H_r^1).$

Proof. To prove that $\mathbf{A} \in \boldsymbol{C}^1 (N_b^*, H_r^1)$, we consider the map $\Psi : N_b^* \times H_r^1 \times \mathbb{R} \to H_r^1 \times \mathbb{R}$, where

$$\Psi(u, v, \sigma) = (v - (-\Delta + 1)^{-1}(\sigma |u|^{p-1} u - \phi_u v), \int |u|^{p-1} u v - c)$$

Then Ψ is of class $\boldsymbol{C}^1$, the implicit function theorem can be applied to Ψ. Note that (2.8) holds if and only if $\Psi(u, v, \sigma) = (0, 0)$. We compute the derivative of Ψ with respect to (v, σ) at the point (u, w, σ) in the direction

$(\bar{w},\bar{\sigma})$ and obtain a map $\Phi: H_r^1\times\mathbb{R} \to H_r^1\times\mathbb{R}$ given by

$$\begin{aligned}\Phi(\bar{w},\bar{\sigma}) &= D_{(v,\sigma)}\Psi(u,w,\sigma)(\bar{w},\bar{\sigma})\\ &=(\bar{w}-(-\Delta+1)^{-1}(\bar{\sigma}|u|^{p-1}u-\phi_u\bar{w}),\int|u|^{p-1}u\bar{w}).\end{aligned}$$

If $\Phi(\bar{w},\bar{\sigma})=(0,0),$ that is

$$-\Delta\bar{w}+\bar{w}+\phi_u\bar{w}=\bar{\sigma}|u|^{p-1}u, \tag{2.9}$$

And $$\int|u|^{p-1}u\bar{w}=0.$$

Multiplying the equation (2.9) by $\bar{w}$ and then integrating it, we get

$$\|\bar{w}\|^2\le\bar{\sigma}\int|u|^{p-1}u\bar{w}=0.$$

Then $\bar{w}=0$ and $\bar{\sigma}|u|^{p-1}u\equiv 0$ in $\mathbb{R}^3$, so $\bar{\sigma}=0$. Hence Φ is injective.

To prove Φ is surjective, given any $(f,c_1)\in H_r^1\times\mathbb{R}$, let $v_1,v_2\in H_r^1$ be solutions of the linear problems

$$\begin{cases}-\Delta v_1+v_1+\phi_u v_1=-\Delta f+f,\\ -\Delta v_2+v_2+\phi_u v_2=|u|^{p-1}u.\end{cases}$$

Since $|u|_{p+1}^{p+1}>\frac{1}{2}c>0,$ so $v_2\ne 0$ and then $\int|u|^{p-1}uv_2>0.$ Let $\bar{\sigma}=\frac{c_1-\int|u|^{p-1}uv_1}{\int|u|^{p-1}uv_2},$ $\bar{w}=v_1+\bar{\sigma}v_2$, then $\Phi(\bar{w},\bar{\sigma})=(f,c_1)$, which implies Φ is surjective. Hence Φ is a bijective map, which implies that $\mathbf{A}\in \boldsymbol{C}^1\,(N_b^*,H_r^1)$. This completes the proof.

Lemma 2.4. Suppose that $\{u_n\}\subset N_b,\ w_n=\mathbf{A}(u_n)$. Then $\{w_n\}$ has a strongly convergent subsequence in H_r^1.

Proof. Let $\{u_n\}\subset N_b,$ then u_n is bounded in H_r^1. By (2.7) and the Sobolev inequality, we have

$$\|\tilde{w}_n\|^2\le\int|u_n|^{p-1}u_n\tilde{w}_n\le c^{\frac{p}{p+1}}|\tilde{w}_n|_{p+1}\le \mathrm{c}_0\|\tilde{w}_n\|.$$

Then $\{w_n\}\subset H_r^1$ is a bounded sequence. Passing to a subsequence, we may assume that $u_n\to u,\ \tilde{w}_n\to\tilde{V}_0$ weakly in H_r^1 and $u_n\to u,\ \tilde{w}_n\to\tilde{V}_0$ strongly in L^s for $s\in(2,6)$. Since $u_n\to u$ strongly in $L^{\frac{12}{5}}(\mathbb{R}^3)$, it follows from Lemma 2.1(6) and the Sobolev imbedding theorem that $\phi_{u_n}\to\phi_u$ strongly in L^6. Consider the identity

$$\int(\nabla\tilde{w}_n\nabla\xi+\tilde{w}_n\xi)+\int\phi_{u_n}\tilde{w}_n\xi=\int|u_n|^{p-1}u_n\xi,\qquad \xi\in H_r^1. \tag{2.10}$$

Using the Hölder inequality, we have

$$\left|\int(\phi_{u_n}\tilde{w}_n\xi-\phi_{u_n}\tilde{V}_0\xi)\right|\le|\phi_{u_n}|_6|\tilde{w}_n-\tilde{V}_0|_{\frac{12}{5}}|\xi|_{\frac{12}{5}}=o(1)$$

for any $\xi\in H_r^1$. Then we get

$$\int\nabla\tilde{w}_n\nabla(\tilde{w}_n-\tilde{V}_0)+\tilde{w}_n(\tilde{w}_n-\tilde{V}_0)=\int\phi_{u_n}\tilde{w}_n(\tilde{w}_n-\tilde{V}_0)+\int|u_n|^{p-1}u_n(\tilde{w}_n-\tilde{V}_0)=o(1).$$

Hence

$$\|\tilde{w}_n\|^2=\int\nabla\tilde{w}_n\nabla\tilde{V}_0+\tilde{w}_n\tilde{V}_0+o(1)=\|\tilde{V}_0\|^2+o(1),$$

which implies $\tilde{w}_n\to\tilde{V}_0$ strongly in H_r^1. Taking limit as $n\to+\infty$ in (2.10) yields

$$\int(\nabla\tilde{V}_0\nabla\xi+\tilde{V}_0\xi)+\int\phi_u\tilde{V}_0\xi=\int|u|^{p-1}u\xi,\qquad \xi\in H_r^1.$$

This implies that $\tilde{V}_0$ satisfies

$$-\Delta\tilde{V}_0+\tilde{V}_0+\phi_u\tilde{V}_0=|u|^{p-1}u.$$

Since $|u|_{p+1}^{p+1}=c$, so $\tilde{V}_0\neq 0$ and then $\int|u|^{p-1}u\tilde{V}_0>0,$ which implies that

$$\lim_{n\to\infty}\sigma_n=\lim_{n\to\infty}\frac{c}{\int|u|^{p-1}u\tilde{w}_n}=\frac{c}{\int|u|^{p-1}u\tilde{V}_0}=\sigma_0.$$

Therefore, $w_n=\sigma_n\tilde{w}_n\to\sigma_0\tilde{V}_0=V_0$ strongly in H_r^1. This completes the proof.

Now let us define a map

$$\mathbf{V}:N_b^*\to H_r^{\mathbf{1}},\quad \mathbf{V}(u)=u-\mathbf{A}(u).$$

To constructing a descending flow for the functional $I(u)$, we prove that $\mathbf{V}$ is a sort of pseudo-gradient vector of $I(u)$ restricted on N_b. We have the following lemma.

Lemma 2.5. $$I'(u)[\mathbf{V}(u)]\geq\|\mathbf{V}(u)\|^2,\qquad \forall\ u\in N_b.$$

Proof. Take any $u\in N_b$ and write $w=\mathbf{A}(u)$ as above. By (2.8), we have $\int|u|^{p-1}u(u-w)=c-c=0.$ Let $v=\mathbf{V}(u)=u-w,$, then $u=v+w$ and $\int|u|^{p-1}uv=0$, we deduce from (2.1) and (2.8) that

$$\begin{aligned}I'(u)[v]&=\int(\nabla u\nabla v+uv)+\int\phi_u uv-\int|u|^{p-\mathbf{1}}uv\\&=\int\nabla(v+w)\nabla v+(v+w)v+\int\phi_u(v+w)v\\&=\|v\|^2+\sigma\int|u|^{p-\mathbf{1}}uv+\int\phi_u v^{\mathbf{2}}\\&\geq\|v\|^{\mathbf{2}}.\end{aligned}$$

Lemma 2.6. Let $u_n\in N_b$ be such that

$$I(u_n)\to d<d_k,\ \text{ and }\ \mathbf{V}(u_n)\to 0\ \text{ strongly in }\ H_r^{\mathbf{1}}.$$

Then, up to a subsequence, there exists $u\in N_b$ such that $u_n\to u$ strongly in $H_r^{\mathbf{1}}$ and $\mathbf{V}(u)=0$.

Proof. Since $u_n\in N_b$, then u_n is bounded. By Lemma 2.4, up to a subsequence, we may assume that $u_n\to u$ weakly in $H_r^{\mathbf{1}}$ and $w_n=\mathbf{A}(u_n)\to V_0$ strongly in $H_r^{\mathbf{1}}$, hence $u_n\to u$ in L^s for $s\in[2,6]$, we have

$$\int u_n\xi\to\int u\xi,\quad \int\nabla(u_n-u)\nabla\xi+(u_n-u)\xi\to 0,\qquad \text{for all}\quad \xi\in H_r^1,$$

and

$$\int|\nabla(w_n-V_0)|^2+|w_n-V_0|^2\to 0.$$

Hence $\int\nabla(u_n-u)\nabla V_0\to 0,\ \int(u_n-u)V_0\to 0,\ \int|\nabla(w_n-V_0)|^2\to 0$ and $\int|w_n-V_0|^2\to 0$. Since $\mathbf{V}(u_n)\to 0$, it reads $\int|\nabla(u_n-w_n)|^2+|u_n-w_n|^2\to 0$, hence $\int|\nabla(u_n-w_n)|^2\to 0$ and $\int|u_n-w_n|^2\to 0$. So we have that

$$\begin{aligned}0\leq\left|\int\nabla u_n\nabla(u_n-u)\right|&=\left|\int\nabla(u_n-w_n+w_n-V_0+V_0)\nabla(u_n-u)\right|\\&=\int|\nabla(u_n-w_n)||\nabla(u_n-u)|+\int|\nabla(w_n-V_0)\nabla(u_n-u)|+\left|\int\nabla V_0\nabla(u_n-u)\right|\\&=c_1\left[\int|\nabla(u_n-w_n)|^2+\int|\nabla(w_n-V_0)|^2\right]+\left|\int\nabla V_0\nabla(u_n-u)\right|=o(1).\end{aligned}$$

Similarly, we have

$$0 \le |\int u_n(u_n-u)| = |\int (u_n-w_n+w_n-V_0+V_0)(u_n-u)|$$
$$\le c_1[\int |u_n-w_n|^2+\int |w_n-V_0|^2]+|\int V_0(u_n-u)|=o(1).$$

Hence $u_n \to u$ strongly in H_r^1 and so $u \in \overline{N_b}$. Therefore, $\mathbf{V}(u)=\lim_{n\to\infty}\mathbf{V}(u_n)=0.$ Moreover, $I(u_n)\to d<d_k$ and so $u \in N_b$. This completes the proof.

To obtain sign-changing solutions, we make use of the positive and negative cones as in many references such as [33, 38]. Precisely, we define

$$P^+=\{u\in H_r^1 : u\ge 0\} \text{ and } P^-=-P^+=\{u\in H_r^1 : u\le 0\}, \text{ set } P=P^+\cup P^-.$$

Moreover, for $\delta>0$ we define $P_\delta=\{u\in H_r^1 : \operatorname{dist}_{p+1}(u,P)<\delta\}$, where

$$\operatorname{dist}_{p+1}(u,P)=\min\{\operatorname{dist}_{p+1}(u,P^+),\quad \operatorname{dist}_{p+1}(u,P^-)\},$$
$$\operatorname{dist}_{p+1}(u,P^\pm)=\inf\{|u-v|_{p+1}:\ v\in P^\pm\}.$$

Denote $u^\pm=\max\{\mathbf{0},\pm u\}$, then $u=u^+-u^-$ and, it is easy to check that $\operatorname{dist}_{p+1}(u,P^\pm)=|u^\mp|_{p+1}$.

Then P_δ is an open and symmetric subset of H_r^1 and $H_r^1\setminus P_\delta$ contains only sign-changing functions.

Lemma 2.7. There exists $\delta_0>0$ such that for $\delta\in(0,\delta_0)$, there holds

$$\operatorname{dist}_{p+1}(\mathbf{A}(u),P)<\frac{1}{2}\delta,\quad \forall u\in N_b,\quad \operatorname{dist}_{p+1}(u,P)<\delta.$$

Proof. For $u\in P_\delta$, we have that $\operatorname{dist}_{p+1}(u,P^+)<\delta$ or $\operatorname{dist}_{p+1}(u,P^-)<\delta$. To show $\operatorname{dist}_{p+1}(\mathbf{A}(u),P)<\frac{1}{2}\delta$, we need to show that either $\operatorname{dist}_{p+1}(\mathbf{A}(u),P^+)<\frac{1}{2}\delta$ or $\operatorname{dist}_{p+1}(\mathbf{A}(u),P^-)<\frac{1}{2}\delta$ be valid. Indeed, for δ small enough, we have the following two statements:

(1) If $\operatorname{dist}_{p+1}(u,P^+)<\delta$, then $\operatorname{dist}_{p+1}(\mathbf{A}(u),P^+)<\frac{1}{2}\delta$.

(2) If $\operatorname{dist}_{p+1}(u,P^-)<\delta$, then $\operatorname{dist}_{p+1}(\mathbf{A}(u),P^-)<\frac{1}{2}\delta$.

Since the two conclusions are similar, it suffices to prove the first one. Let $w=\mathbf{A}(u)=w^+-w^-$, we have

$$\operatorname{dist}_{p+1}(w,P^+)\|w^-\|=|w^-|_{p+1}\|w^-\|\le c_0\|w^-\|=-c_0(w,w^-)_{H_r^1}$$
$$=-c_0[\int |u|^{p-1}uw^- - \int \phi_u ww^-]$$
$$=-c_0[\int |u^+|^p w^- - \int |u^-|^p w^- + \int \phi_u (w^-)^2]$$
$$\le c_0\int |u^-|^p w^- \le c_0|u^-|_{p+1}^p\|w^-\|.$$

Therefore

$$\operatorname{dist}_{p+1}(w,P^+)\le c_0|u^-|_{p+1}^{p-1}|u^-|_{p+1}=c_0|u^-|_{p+1}^{p-1}\operatorname{dist}_{p+1}(u,P^+).$$

Let $\delta_0>0$ small enough such that $c_0\delta_0^{p-1}<\frac{1}{2}.$ Then we get that, if $\delta\in(0,\delta_0)$ and $u\in N_b$ with $\operatorname{dist}_{p+1}(u,P^+)<\delta$, we have that $\operatorname{dist}_{p+1}(\mathbf{A}(u),P^+)<\frac{1}{2}\delta$. This completes the proof.

To continue our proof, we introduce a notion of local genus simulating that of vector genus introduced by [38] to define suitable minimax energy levels. To do this, we consider the class of sets

$$\boldsymbol{F} = \{B \subset M: B \text{ is closed and symmetric with respect to } 0\},$$

and, for each $B \in \boldsymbol{F}$ and $k \in \mathrm{N}$, the class of functions

$$F_k(B) = \{f : B \to \mathbb{R}^{k-1}, f \text{ is odd and } f \in \boldsymbol{C}(B, \mathbb{R}^{k-1})\}.$$

Here we denote $\mathrm{R}^0 = \{0\}$. The genus γ of $B \in \boldsymbol{F}$ is a number in $\mathbb{N} \cup \{+\infty\}$. We say that $\gamma(B) \geq k$ if for every $f \in F_k(B)$ there exists $u \in B$ such that $f(u) = 0$. We denote

$$\Gamma_k = \{B \in \boldsymbol{F} :: \gamma(B) \geq k\}.$$

As usual, we have the following useful properties of the genus.

Lemma 2.8.

(1) Let $B \subset M$ and let η: $S^{k-1} = \{x \in \mathbb{R}^k, |x| = c, c > 0\} \to B$ be an odd homeomorphism. Then $B \in \Gamma_k$.

(2) There holds $\overline{\eta(B)} \in \Gamma_k$ whenever $B \in \Gamma_k$ and $\eta : B \to M$ is a continuous odd map.

The following two lemmas are crucial in constructing suitable minimax values of I.

Lemma 2.9. Let $k \geq 2$. Then there exists $\delta_0 > 0$, for any $\delta \in (0, \delta_0)$ and any $B \in \Gamma_k$, there holds $B \setminus P_\delta \neq \varnothing$.

Proof. For any $B \in \Gamma_k$. By the definition of Γ_k, then for any $f \in F_k(B)$ there exists $u \in B$ such that $f(u) = 0$. Consider the function $B \to \mathbb{R}^{k-1}$ defined as $f(u) = (\int |u|^p u, 0, \cdots, 0) \in \mathbb{R}^{k-1}$. Clearly $f \in F_k(B)$, so there exists $u \in B$ such that $f(u) = 0$. Note that $u \in M$, that is $\int |u|^{p+1} = c$, we conclude that

$$\int |u^+|^{p+1} = \int |u^-|^{p+1} = \frac{1}{2}c,$$

that is, $\mathrm{dist}_{p+1}(u, P) = (\frac{1}{2}c)^{\frac{1}{p+1}}$, and so $u \in B \setminus P_\delta$ for every $\delta < \delta_0 \leq (\frac{1}{2}c)^{\frac{1}{p+1}}$.

Lemma 2.10. There exists $B \in \Gamma_{k+1}$ such that $B \subset N_b$ and $\sup_{u \in B} I(u) < d_k$.

Proof. Let W_{k+1} be a $k+1$ dimensional subspace of H_r^1. We define $B = B_k = \{u \in W_{k+1}: |u|_{p+1}^{p+1} = c\}$. Obviously, there exists an odd homeomorphism from S^k to B. By Lemma 2.8 (1) one has $B \in \Gamma_{k+1}$. From (2.2) we have that $B \subset N_{b_k}$, and so Lemma 2.2 yields $\sup_{u \in B} I(u) < d_k$.

Now we are in a position to construct the minimax values for I. For every $k_1 \in [2, k+1]$ and $\delta < \delta_0 \leq (\frac{1}{2}c)^{\frac{1}{p+1}}$, we define

$$c_{\delta k_1} = \inf_{B \in \Gamma_{k_1}^0} \sup_{u \in B \setminus P_\delta} I(u), \tag{2.11}$$

where $$\Gamma_{k_1}^0 = \{B \in \Gamma_{k_1} : B \subset N_b \text{ and } \sup_B I < d_k\}.$$

Note that $\Gamma_{k_2}^0 \subset \Gamma_{k_1}^0$ for any $k_2 \geq k_1$, hence $\Gamma_{k_1}^0 \neq \varnothing$ and so $c_{\delta k_1}$ is well defined for any $k_1 \in [2, k+1]$. Moreover, $c_{\delta k_1} < d_k$ for every $\delta \in (0, \delta_0)$ and $k_1 \in [2, k+1]$. Define $N_b^0 = \{u \in N_b : I(u) < d_k\}$, then by Lemma 2.2 $B_k \subset N_b^0$.

Now we can construct a descending flow for the functional I, and then the set N_b^0 will be seen turned out to be the desired invariant set of the flow.

Lemma 2.11. There exists a unique global solution $\eta : [0, +\infty) \times N_b^0 \to H_r^1$ for the initial value problem

$$\frac{d\eta(t,u)}{dt} = -\mathbf{V}(\eta(t,u)), \quad \eta(0,u) = u \in N_b^0, \tag{2.12}$$

which satisfies

(1) $\eta(t,u) \in N_b^0$ for any $t>0$ and $u \in N_b^0$.

(2) $\eta(t,-u) = -\eta(t,u)$ for any $t>0$ and $u \in N_b^0$.

(3) For every $u \in N_b^0$, the map $t \to I(\eta(t,u))$ is non-increasing.

(4) There exists $\delta_0 \in (0, (\frac{1}{2}c)^{\frac{1}{p+1}})$ such that, for every $\delta < \delta_0$, there holds

$$\eta(t,u) \in P_\delta \quad \text{whenever} \quad u \in N_b^0 \cap P_\delta \quad \text{and} \quad t>0.$$

Proof. The proof is similar to that has shown as in [39]. For the sake of completeness we reproduce that proof here.

Recalling Lemma 2.3, it shows that $\mathbf{V}(u) \in \mathbf{C}^1(N_b^*, H_r^1)$. Since $N_b^0 \subset N_b^*$ and N_b^* be open, so (2.12) admits a unique solution $\eta(t,u) \in N_b^*$, where $T_{\max} > 0$ is the maximal time such that $\eta : [0, T_{\max}) \times N_b^0 \to N_b^* \subset H_r^1$ for all $t \in [0, T_{\max})$ (since $\mathbf{V}(u)$ is defined only on N_b^*). We should prove $T_{\max} = +\infty$ for any $u \in N_b^0$. Reasoning by contradiction, suppose that there exists some $u_0 \in N_b^0$, the flow starting from which the maximal time $T_{\max} < +\infty$. Consider

$$\begin{aligned}\frac{d}{dt}\int |\eta(t,u_0)|^{p+1} &= -(p+1)\int |\eta(t,u_0)|^{p-1}\, \eta(t,u_0)(\eta(t,u_0) - \mathbf{A}(\eta(t,u_0))) \\ &= (p+1)c - (p+1)\int |\eta(t,u_0)|^{p+1}, \quad \forall\ 0<t<T_{\max}.\end{aligned}$$

Since $\int |\eta(0,u_0)|^{p+1} = \int |u_0|^{p+1} = c$, we infer that $\int |\eta(t,u_0)|^{p+1} \equiv c$ for all $0 \le t < T_{\max}$. Then $\eta(t,u_0) \in M \cap N_b^* = N_b$ for all $t \in [0, T_{\max})$, hence $\eta(T_{\max}, u_0) \in \partial N_b$, and so $I(\eta(T_{\max}, u_0)) \ge d_k$. Since $\eta(t,u_0) \in N_b$ for all $t \in [0, T_{\max})$, we deduce from Lemma 2.5 that

$$\begin{aligned} I(\eta(T_{\max},u_0)) &= I(u_0) - \int_0^{T_{\max}} I'(\eta(t,u_0))[\mathbf{V}(\eta(t,u_0))]dt \\ &\le I(u_0) - \int_0^{T_{\max}} \|\mathbf{V}(\eta(t,u_0))\|^2 dt \le I(u_0) < d_k, \end{aligned}$$

a contradiction. So $T_{\max} = +\infty$, and above inequality shows similarly that $I(\eta(t,u)) \le I(u) < d_k$ for all $t>0$ and $u \in N_b^0$, hence previous argument shows that $\eta(t,u) \in N_b^0$ for all $t>0$ and then (1), (2), (3) hold.

Finally, let $\delta_0 \in (0, (\frac{1}{2}c)^{\frac{1}{p+1}})$ be such that Lemma 2.7 holds for $\delta < \delta_0$. For any $u \in N_b^0$ with $\mathrm{dist}_{p+1}(u, P) \le \delta < \delta_0$, since

$$\eta(t,u) = u + t\frac{d}{dt}\eta(0,u) + o(t) = (1-t)u + t\mathbf{A}(u) + o(t),$$

we achieve that

$$\begin{aligned}\mathrm{dist}_{p+1}(\eta(t,u),P) &= \mathrm{dist}_{p+1}((1-t)u + t\mathbf{A}(u) + o(t), P) \\ &\le (1-t)\,\mathrm{dist}_{p+1}(u,P) + t\,\mathrm{dist}_{p+1}(\mathbf{A}(u),P) + o(t) \\ &\le (1-t)\delta + \frac{1}{2}t\delta + o(t) < \delta\end{aligned}$$

for $t>0$ small enough. Hence (4) holds.

3. PROOF OF THEOROM 1.1

After all the preparations above, now we are in a position to prove Theorem 1.1.

Proof. **of Theorem 1.1.(Existence part)** Take any $k_1 \in [2, k+1]$ and $\delta \in (0, \delta_0)$, write $d = c_{\delta k_1}$ for convenience in this part. We prove that there exists a couple (u_c, λ_c) with u_c changing its sign and $|u_c|_{p+1}^{p+1} = c$ such that (u_c, λ_c) is a solution to (1.1), that is, $d = c_{\delta k_1}$ is a correspondent value of some critical value of I_{λ_c}.

We claim that there exists a sequence $\{u_n\} \subset N_b^0$ such that

$$I(u_n) \to d,\ \ \mathbf{V}(u_n) \to 0 \ \text{ as } \ n \to +\infty, \ \text{ and } \ \mathrm{dist}_{p+1}(u_n, P) \geq \delta, \quad \forall n \in \mathbb{N}^+. \qquad (3.1)$$

Proving this claim by contradiction. Suppose that (3.1) does not hold, recalling that $d < d_k$, there exists small $\varepsilon \in (0,1)$ such that

$$\|\mathbf{V}(u)\|^2 \geq \varepsilon, \ \ \forall u \in N_b^0, \ \ |I(u) - d| \leq 2\varepsilon, \ \ \mathrm{dist}_{p+1}(u, P) \geq \delta.$$

Recalling the definition of $d = c_{\delta k_1}$ in (2.11), we see that there exists $B \in \Gamma_{k_1}^0$ such that

$$\sup_{u \in B \setminus P_\delta} I(u) < d + \varepsilon.$$

Noting that $B \subset N_b^0$, we can consider $D = \eta(2, B)$, where η is in Lemma 2.11. Hence $D \subset N_b^0$. Lemma 2.8 (2) and Lemma 2.11 (2) imply that $D \in \Gamma_{k_1}$. By Lemma 2.11 (3), we have $\sup_D I \leq \sup_B I < d_k$, that is $D \in \Gamma_{k_1}^0$ and so $\sup_{D \setminus P_\delta} I \geq d$. Let $u_1 \in D \setminus P_\delta$ such that $\sup_{D \setminus P_\delta} I - \varepsilon < I(u_1)$, then there exists $u \in B$ such that $\eta(2, u) = u_1$ and

$$d - \varepsilon \leq \sup_{D \setminus P_\delta} I - \varepsilon < I(u_1) = I(\eta(2, u)).$$

Since $\eta(t, u) \in N_b^0$ for any $t \geq 0$ and $\eta(2, u) = u_1 \notin P_\delta$, Lemma 2.11 (4) shows that $\eta(t, u) \notin P_\delta$ for all $t \in [0, 2]$. In particular, $u \notin P_\delta$ and so $I(u) < d + \varepsilon$. Hence for all $t \in [0, 2]$ we have

$$d - \varepsilon < I(\eta(2, u)) \leq I(\eta(t, u)) \leq I(u) < d + \varepsilon.$$

Which deduces $\|\mathbf{V}(\eta(t, u))\|^2 \geq \varepsilon$ and

$$\frac{d}{dt} I(\eta(t, u)) = -I'(\eta(t, u))[\mathbf{V}(\eta(t, u))] \leq -\|\mathbf{V}(\eta(t, u_0))\|^2 \leq -\varepsilon,$$

for every $t \in [0, 2]$. Therefore, we arrive at

$$d - \varepsilon < I(\eta(2, u)) \leq I(u) - \int_0^2 \varepsilon dt < d + \varepsilon - 2\varepsilon = d - \varepsilon,$$

a contradiction. Then (3.1) holds. By Lemma 2.6, up to a subsequence, there exists $u \in N_b$ such that $u_n \to u$ strongly in H_r^1 and $\mathbf{V}(u) = 0, I(u) = d = c_{\delta k_1}$. Since $\mathbf{V}(u) = u - \mathbf{A}(u) = 0$, that is $u = \mathbf{A}(u)$, hence u satisfies

$$\begin{cases} -\Delta u + u + \phi_u u = \sigma |u|^{p-1} u, \\ \int |u|^{p+1} = c. \end{cases}$$

Since $\mathrm{dist}_{p+1}(u, P) \geq \delta$, we know that $u \notin P_\delta$, hence u is sign-changing. Let

$$\lambda_c = \sigma = \frac{\|u\|^2 + \int \phi_u |u|^2}{c}, \quad u_c = u.$$

We see that (u_c, λ_c) solves the problem $(\mathrm{P_c})$. In a word, for any $k_1 \in [2, k+1]$, every $c_{\delta k_1}$ corresponds to a critical value of I_λ such that $I_{\lambda_c}(u_c) = c_{\delta k_1} + \frac{c}{p+1}(1 - \lambda_c)$ for some couple (u_c, λ_c) which solves the problem $(\mathrm{P_c})$.

(Multiplicity part) We prove that system (1.1) has infinitely many sign-changing normalized solutions. Reasoning by contradiction, suppose that there exists $n_0 \in \mathbb{N}^+$ such that system (1.1) has only n_0 such solutions. Take $k \geq n_0 + 1$ fixed and $\delta \in (0, \delta_0)$, since $\Gamma^0_{k_1+1} \subset \Gamma^0_{k_1}$, we have

$$c_{\delta 2} \leq c_{\delta 3} \leq \cdots \leq c_{\delta k} \leq c_{\delta(k+1)} < d_k. \tag{3.2}$$

Since $c_{\delta k_1}$ are correspondent values of critical values of I_λ for all $k_1 \in [2, k+1]$ with some couple (u_c, λ_c). We show that for any two different minimax values $c_{\delta k_1}$, the corresponding couples (u_c, λ_c) are different. Set $d_1 \neq d_2$ are two such values, d_i corresponds to the couple (u_i, λ_i). If $I_{\lambda_1}(u_1) \neq I_{\lambda_2}(u_2)$, then $(u_1, \lambda_1) \neq (u_2, \lambda_2)$ obviously. If $I_{\lambda_1}(u_1) = I_{\lambda_2}(u_2)$, since $I_{\lambda_i}(u_i) = d_i + \frac{c}{p+1}(1-\lambda_i)$, one has $\lambda_1 - \lambda_2 = \frac{p+1}{c}(d_2 - d_1) \neq 0$, then $\lambda_1 \neq \lambda_2$ and so $u_1 \neq u_2$, hence $(u_1, \lambda_1) \neq (u_2, \lambda_2)$. Therefore, there certainly exists some $2 \leq N_1 \leq k$ such that

$$c_{\delta N_1} = c_{\delta(N_1+1)} = \bar{c} < d_k. \tag{3.3}$$

Define

$$\boldsymbol{K} = \{u \in N_b : u \text{ is sign-changing}, \quad I(u) = \bar{c} \quad \text{and} \quad \mathbf{V}(u) = 0\}. \tag{3.4}$$

Then $\boldsymbol{K}$ is finite and symmetric, that is, $\boldsymbol{K} \in \boldsymbol{F}$. Then there exists $k_0 \leq k-1$ and $\{u_m : 1 \leq m \leq k_0\} \subset \boldsymbol{K}$ such that

$$\boldsymbol{K} = \{u_m, \ -u_m : 1 \leq m \leq k_0\}.$$

Taking $\boldsymbol{O}_{u_m}$ be open neighborhoods of u_m in H, such that any two of $\overline{\boldsymbol{O}_{u_m}}$ and $-\overline{\boldsymbol{O}_{u_m}}$, where $1 \leq m \leq k_0$, are disjoint and

$$\boldsymbol{K} \subset \boldsymbol{O} = \bigcup_{m=1}^{k_0} \overline{\boldsymbol{O}_{u_m}} \cup -\overline{\boldsymbol{O}_{u_m}}.$$

Define a continuous map $\tilde{f} : \boldsymbol{O} \to \mathbb{R} \setminus \{0\}$ by

$$\tilde{f}(u) = \begin{cases} 1, & \text{if} \quad u \in \bigcup_{m=1}^{k_0} \overline{\boldsymbol{O}_{u_m}}, \\ -1, & \text{if} \quad u \in \bigcup_{m=1}^{k_0} -\overline{\boldsymbol{O}_{u_m}}. \end{cases}$$

Then $\tilde{f}(-u) = -\tilde{f}(u)$. Then by Tietze's extension theorem, there exists $f \in \boldsymbol{C}(H, \mathbb{R})$ such that $f|_{\boldsymbol{O}} \equiv \tilde{f}$. Define

$$F(u) = \frac{f(u) - f(-u)}{2},$$

then $F|_{\boldsymbol{O}} \equiv \tilde{f}$ and, is odd on H. Define

$$\boldsymbol{K}_\tau = \{u \in N_b : \inf_{v \in \boldsymbol{K}} \|u - v\| < \tau\}.$$

Take $\tau > 0$ small such that $\boldsymbol{K}_{2\tau} \subset \boldsymbol{O}$. Recalling $\mathbf{V}(u) = 0$ in $\boldsymbol{K}$ and $\boldsymbol{K}$ is finite, there exists $C > 0$ such that

$$\|\mathbf{V}(u)\| \leq C, \qquad \forall\, u \in \overline{\boldsymbol{K}_{2\tau}}. \tag{3.5}$$

By (3.4) and Lemma 2.6, it is easy to see that there exists small $\varepsilon \in (0, \frac{d_k - \bar{c}}{2})$ such that

$$\|\mathbf{V}(u)\|^2 \geq \varepsilon, \quad \forall u \in N_b \setminus (\boldsymbol{K}_\tau \cup P_\delta) \quad \text{satisfying} \quad |I(u) - d| \leq 2\varepsilon. \tag{3.6}$$

Let $\alpha = \frac{1}{2}\min\{1, \frac{\tau}{3C}\}$. Then we can take $B \in \Gamma^{\mathbf{0}}_{N_1+1}$ such that

$$\sup_{B \setminus P_\delta} I < c_{\delta(N_1+1)} + \alpha\varepsilon = \bar{c} + \alpha\varepsilon. \tag{3.7}$$

Let $D = B \setminus \boldsymbol{K}_{2\tau}$, then $D \in \boldsymbol{F}$. We claim that $\gamma(D) \geq N_1$. Otherwise, then there exists $\tilde{g} \in F_{N_1}(D)$ such that for any $u \in D, \tilde{g} \neq 0$. By Tietze's extension theorem, we get a map $\bar{g} \in \boldsymbol{C}(H, \mathbb{R}^{N_1-1})$ such that $\bar{g}|_D \equiv \tilde{g}$. Define

$$g(u)=\frac{\bar{g}(u)-\bar{g}(-u)}{2}, \qquad \forall\ u\in H,$$

then $g|_D\equiv\tilde{g}$ and is odd. Let $G(u)=(g(u),F(u))$ for $u\in B$, then $G\in C(H,\mathbb{R}^{N_1-1+1})$ and is odd. Hence $G\in F_{N_1+1}(B)$. Since $B\in\Gamma_{N_1+1}$, so $G(u)=(g(u),F(u))=0$ for some $u\in B$. If $u\in K_{2\tau}\subset O$, then $F(u)=0$, a contradiction. So $u\in B\backslash K_{2\tau}=D$, and then $0=g(u)=\tilde{g}(u)\neq 0$, also a contradiction. Hence $\gamma(D)\geq N_1$, that is, $D\in\Gamma_{N_1}$. Note that $D\subset B\subset N_b$ and $B\in\Gamma^{\mathbf{0}}_{N_1+1}$, then $\sup_D I\leq\sup_B I<d_k$, we obtain that $D\subset N_b^{\mathbf{0}}$ and $D\in\Gamma^{\mathbf{0}}_{N_1}$. We consider $E=\eta(\frac{\tau}{3C},D)$. As previous proof in existence part, we have $E\in\Gamma^{\mathbf{0}}_{N_1}$, hence $\sup_{E\backslash P_\delta} I\geq c_{\delta N_1}=\bar{c}$. On the other hand, there exists $u_1\in E\setminus P_\delta$ such that $\sup_{E\backslash P_\delta} I-\alpha\varepsilon<I(u_1)$, hence there exists $u\in D$ such that $\eta(\frac{\tau}{3C},u)=u_1$ and then, we have

$$\bar{c}-\alpha\varepsilon\leq\sup_{E\backslash P_\delta} I-\alpha\varepsilon<I(u_1)=I(\eta(\frac{\tau}{3C},u)).$$

Since $\eta(t,u)\in N_b^0$ for all $t\geq 0$ and $\eta(\frac{\tau}{3C},u)=u_1\notin P_\delta$, we have $\eta(t,u)\notin P_\delta$ for all $t\in[0,\frac{\tau}{3C}]$. In particular, $u\notin P_\delta$ and so $I(u)<\bar{c}+\alpha\varepsilon$ by (3.7), since $u\in D=B\backslash K_{2\tau}\subset B$. Then for any $t\in[0,\frac{\tau}{3C}]$, we have

$$\bar{c}-\alpha\varepsilon<I(\eta(\frac{\tau}{3C},u))\leq I(\eta(t,u))\leq I(u)<\bar{c}+\alpha\varepsilon.$$

In order to use (3.6), we need to show that $\eta(t,u)\notin K_\tau$ for all $t\in[0,\frac{\tau}{3C}]$. If there exists $T\in[0,\frac{\tau}{3C}]$ such that $\eta(T,u)\in K_\tau$, then there exist $0\leq t_1<t_2\leq T$ such that $\eta(t_1,u)\in\partial K_{2\tau},\eta(t_2,u)\in\partial K_\tau$, and $\eta(t,u)\in K_{2\tau}\setminus K_\tau$ for $t\in(t_1,t_2)$. So we see from (3.5) that

$$\tau\leq\|\eta(t_1,u)-\eta(t_2,u)\|=\|\int_{t_1}^{t_2}\mathbf{V}(\eta(t,u)dt\|\leq 2C(t_2-t_1),$$

that is, $\frac{\tau}{2C}\leq t_2-t_1\leq T\leq\frac{\tau}{3C}$, a contradiction. Hence $\eta(t,u)\notin K_\tau$ for all $t\in[0,\frac{\tau}{3C}]$, hence $\|\mathbf{V}(\eta(t,u))\|^2\geq\varepsilon$ and, we achieve that

$$\bar{c}-\alpha\varepsilon<I(\eta(\frac{\tau}{3C},u))\leq I(u)-\int_0^{\frac{\tau}{3C}}\varepsilon dt<\bar{c}+\alpha\varepsilon-2\alpha\varepsilon=\bar{c}-\alpha\varepsilon,$$

a contradiction. Hence we have infinitely many different values of $c_{\delta(k+1)}$. This completes the proof.

4. PROOF OF THEOROM 1.2

Proof. **of Theorem 1.2.** Define

$$K_c=\{(u_c,\lambda_c):\ (u_c,\lambda_c)\text{ solves the problem }(\mathrm{P_c})\text{ with }u_c\text{ sign-changing}\}.$$

Then $K_c\neq\varnothing$. Let $d=\inf_{(u_c,\lambda_c)\in K_c} I_\lambda(u)$. Then d is well defined since $I_\lambda(u)\geq\frac{1}{4}\|u\|^2$ for $u\in N_\lambda$, the Nehari manifold defined as $N_\lambda=\{(u,\lambda)\in H\setminus\{0\}\times\mathbb{R}^+:I'_\lambda(u)[u]=0\}$. Take $k=1$ in Section 3, (1.1) has a couple (u_c,λ_c) with $I_{\lambda_c}(u_c)=c_{\delta 2}<d_1$ solving the problem $(\mathrm{P_c})$. Hence $d<d_1$. Let $(u_c^n,\lambda_c^n)=(u_n,\lambda_n)\in K_c$ be a minimizing sequence of d with $I_{\lambda_n}(u_n)<d_1$ for all $n\geq 1$, then $\|u_n\|^2\leq 4I_{\lambda_n}(u_n)<4d_1$, that is, $\{u_n\}$ is a bounded sequence. Since (u_n,λ_n) solves $(\mathrm{P_c})$, we have $I'_{\lambda_n}(u_n)=0$ and $\lambda_n=\frac{\|u_n\|^2+\int\phi_{u_n}|u_n|^2}{c}<c_0$, then λ_n has a convergent subsequence which

still likewise labeled as λ_n and set $\lambda_n \to \lambda_c = \lambda$. Recalling the Sobolev inequality $S|u_n|_{p+1}^2 \le \|u_n\|^2$ and $(u_n, \lambda_n) \in \boldsymbol{K}_c$, we deduce that $\inf\{\lambda_n, \lambda\} \ge S\, c^{\frac{p-1}{p+1}} > 0$. Then we have

$$I'_\lambda(u_n)[v] = \int \nabla u_n \nabla v + u_n v + \phi_{u_n} u_n v - \lambda |u_n|^{p-1} u_n v$$

and

$$I'_{\lambda_n}(u_n)[v] = 0 = \int \nabla u_n \nabla v + u_n v + \phi_{u_n} u_n v - \lambda_n |u_n|^{p-1} u_n v.$$

Then we evaluate that

$$I'_\lambda(u_n)[v] = (\lambda_n - \lambda)\int |u_n|^{p-1} u_n v$$

which implies that

$$|I'_\lambda(u_n)[v]| \le |\lambda_n - \lambda| \ \left|\int |u_n|^{p-1} u_n v\right|.$$

Recalling the Sobolev inequality $S|v|_{p+1}^2 \le \|v\|^2$ again, we deduce that

$$|I'_\lambda(u_n)[v]| \le c_1 |\lambda_n - \lambda| \ |u_n|_{p+1}^p \|v\|,$$

that is, $|I'_\lambda(u_n)| \le c_1 |\lambda_n - \lambda|$ which implies that $I'_\lambda(u_n) \to 0$. Hence $\{u_n\}$ is a PS sequence of I_λ, and the fact that the PS condition is valid for $p \in [3,5)$ and for any $\lambda > 0$ has been inferred in [6]. Then u_n has a strongly convergent subsequence which still likewise labeled as u_n. Suppose $u_n \to u$ strongly in H with $|u|_{p+1}^{p+1} = c$, then $I'_\lambda(u) = 0$ and $I_\lambda(u) = d$. We need to show u changing sign. Recalling the Sobolev inequality $S|u|_{p+1}^2 \le \|u\|^2$, we deduce from $I'_{\lambda_n}(u_n)[u_n^\pm] = 0$ that

$$S|u_n^\pm|_{p+1}^2 \le \|u_n^\pm\|^2 = \lambda_n |u_n^\pm|_{p+1}^{p+1} - \int \phi_{u_n} |u_n^\pm|^2 \le c_0 |u_n^\pm|_{p+1}^{p+1},$$

which implies that $|u_n^\pm|_{p+1} \ge (\frac{S}{c_0})^{\frac{1}{p-1}} = c_1 > 0$ for all $n \ge 1$. Hence $|u^\pm|_{p+1} \ge c_1$ and so $(u, \lambda) \in \boldsymbol{K}_c$. This completes the proof.

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